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The isochronous modified Emden oscillators through commutative factorization

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Abstract

In this paper, we focus on the nonlinear oscillators of the modified Emden class of odd positive power $2q+1$ *with power* q *damping coefficients and additional linear harmonic term. They form an interesting type of Liénard oscillators that we approach by the generalized commutative factorization method in which isochronous solutions are obtained from equivalent Bernoulli equations. The first four isochronous oscillators in this class are discussed in some detail. These results show that there exists an infinite family of isochronous oscillators of this type, despite having power* q *damping coefficients.*

Liénard equation; commutative factorization; Bernoulli equation; isochronous waveform..

1 Introduction

The modified Emden class of nonlinear oscillators of arbitrary power q with additional linear term of harmonic type (MEOLHT) are defined by the following differential equations

$$x'' + (q+2)kx^q x' + k^2 x^{2q+1} + \omega^2 x = 0 \, , \tag{1}$$

where q is a positive integer, $q \geq 0$, and k and ω^2 are arbitrary real constants. They are also Liénard nonlinear differential equations

$$x'' + f(x)x' + g(x) = 0 \, , \tag{2}$$

with $f(x)$ and $g(x)$ given by

$$f(x) = (q+2)kx^q \quad \text{and} \quad g(x) = k^2 x^{2q+1} + \omega^2 x \, . \tag{3}$$

The special case $\omega^2 = 0$ and $q = 1$ of (1), i.e., $x'' + 3kxx' + k^2x^3 = 0$ is known to be related to so-called modified Lane-Emden equations in astrophysics, see e.g., [1].

The modification with respect to the original non linear Lane-Emden equations [2, 3] resides only in the coefficient of the first derivative term, there still of linear form, due to a more general type of hydrostatic equilibrium in astrophysical gas systems such as polytropic stars. For more recent works in this area, see [4, 5] and references therein. Recently, modified (generalized) Lane-Emden equations with $\omega^2 \neq 0$ as a plasma frequency have been discussed in the case of atomic clouds confined in magneto-optical traps [6, 7]. For all these equations, the independent variable is a non dimensional radius, while the dependent variable is a non dimensional particle density.

The modified Emden equation as it stands in (1), but for $q = 1$, has been first analysed by Chandrasekar et al. [8] in 2005, who considered it as a Liénard type nonlinear oscillator assuming a temporal variable t as the independent variable. They noticed its intriguing properties such as the frequency completely independent of amplitude (isochronism), just as in the case of linear harmonic oscillator, and its conservative Hamiltonian behavior despite the presence of the nonlinear dissipative term. In another important paper, Iacono and Russo [9] studied the same equation for the odd power case $q = 2m+1$, $m \in \mathbb{N}$, and $\omega = 1$ by an approach involving an integrability condition due to Sabatini [10], see also [11]. Very recently, the case $q = 1$ of (1) has been derived from an equation of Levinson-Smith kind and also its bi-Hamiltonian character has been mentioned in [12].

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In addition, Ramya Parkavi et al. [13] obtained Bernoulli type solutions of the isochronous coupled system of two nonlinear oscillators given by

$$\left\{\begin{array}{ll} \dot{x} & = \omega y - (a_1 x^q + a_2 y^q)x \ , \\ \dot{y} & = -\omega x - (a_1 x^q + a_2 y^q)y \ , \end{array}\right. \tag{4}$$

where q is a positive integer and a_1 and a_2 are constants.

On the other hand, many Liénard equations, in particular those with polynomial coefficients, such as (1), can be conveniently factorized in the form

$$[D - \phi_2(x)][D - \phi_1(x)]x = 0 \ , \quad D = \frac{d}{dt} \ , \tag{5}$$

if the factorization functions ϕ_i satisfy the conditions [14, 15]

$$\phi_2 + \tfrac{d(\phi_1 x)}{dx} = -f(x) \ , \tag{6}$$

$$\phi_1 \phi_2 x = g(x) \ . \tag{7}$$

The commutative case occurs when the factorization functions are of the form $\phi_1 = \phi + c$ and $\phi_2 = \phi - c$ that allows the reverting of the factorization brackets without changing the equation [16].

In the recent paper [17], we showed that a slightly generalized form of the commutative factorization described in [16] provides the general solutions of (1) as solutions of an associated Bernoulli equation and the case $q = 1$ has been briefly presented in this approach. Here, we study these oscillators in the general isochronous case $q = 2n + 1$, with $n \geq 0$ a positive integer, using this generalized commutative factorization approach and the associated Bernoulli trigonometric equation that have the same solutions as developed by us in [17].

2 The q-MEOLHT Equations

The lowest order nonlinear ODE in the MEOLHT class, $q = 1$ in (1),

$$\ddot{x} + 3kx\dot{x} + k^2x^3 + \omega^2 x = 0 \ , \tag{8}$$

has been already discussed in the factorization approach in [17]. It admits the commutative factorization

$$\left(\frac{d}{dt} + kx + i\omega\right)\left(\frac{d}{dt} + kx - i\omega\right)x = 0 \ , \tag{9}$$

where $\phi_1 = -kx + i\omega$, $\phi_2 = -kx - i\omega$, and $c = i\omega$ is an imaginary constant. For this case, the equivalent Bernoulli equation is

$$\dot{x} + \omega \tan(\omega t + \delta)x = -kx^2 \ , \tag{10}$$

where the phase parameter δ is one of the integration constants of the general solution for any q case. Henceforth we set $\delta = 0$ to simplify the notation without much loss in the generality of the results. For $q > 1$ in (1), one can use the factorization brackets with higher order monomials

$$\left(\frac{d}{dt} + kx^q + i\omega\right)\left(\frac{d}{dt} + kx^q - i\omega\right)x = 0 \ , \tag{11}$$

which is also a natural generalization of the $q = 1$ case.

Equation (1) can be converted to a system of first order differential equations. Since we deal with Liénard equations, the most convenient form of the dynamical system is given by

$$\left\{\begin{array}{ll} \dot{x} & = -\int^x f(\xi)d\xi + y \ , \\ \dot{y} & = -g(x) \ , \end{array}\right. \tag{12}$$

or, explicitly, using (3)

$$\left\{\begin{array}{ll} \dot{x} & = -k\frac{q+2}{q+1}x^{q+1} + y \ , \\ \dot{y} & = -k^2x^{2q+1} - \omega^2 x \ . \end{array}\right. \tag{13}$$

It is immediately seen that $x = 0$, $y = 0$, also meaning $x = 0$, $\dot{x} = 0$, is a possible equilibrium point, so we evaluate the Jacobian matrix at the origin as

$$J = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \tag{14}$$

with characteristic equation $\lambda^2 + \omega^2 = 0$. Thus, $\lambda_{1,2} = \pm i\omega$ which defines a stable center at the origin. This can be noticed in the phase portraits presented next in this work.

The main point in the following is that the method described in our paper [17] provides the solution for any q through the reduction to the Bernoulli equation

$$\dot{x} + \omega \tan(\omega t + \delta)x = -kx^{q+1} \, . \tag{15}$$

Taking $u = x^{-q}$, i.e., $x = u^{-1/q}$ and $\dot{x} = -\frac{1}{q}u^{-(q+1)/q}\dot{u}$, turns (15) into the linear equation

$$\dot{u} - q\omega \tan(\omega t + \delta)u = qk \, , \tag{16}$$

which leads straightforwardly to solutions x of the form

$$x(t) = \frac{\cos(\omega t + \delta)}{\left(c_1 + qk \int^t \cos^q(\omega\tau + \delta)d\tau\right)^{1/q}} \tag{17}$$

that may be called Bernoulli type solutions.

If $q \in \mathbb{N}$ is an odd number, we have the following general solution [18]

$$x(t) = \frac{\cos(\omega t + \delta)}{\left(c_1 + q\frac{k}{\omega}\frac{1}{2^{2n}}\sum_{j=0}^{n}\binom{2n+1}{j}\frac{\sin((2n-2j+1)\omega t+\delta)}{2n-2j+1}\right)^{1/q}} \, , \quad q = 2n + 1 \, , \ n \in \mathbb{N} \, , \tag{18}$$

The period for the odd cases has been computed applying the translation operator to the function $x(t)$, $\hat{T}x(t) = x(t + T)$. Using this displacement and some trigonometric identities, we find that $T = 2\pi/\omega$ is the same for all q and independent of the value of c_1. This fact can be illustrated through phase portraits depicting energy sets that correspond to given values of the energy. In general, if the period is the same, independent of the chosen energy, we say that the system is isochronous.

Due to its rational form, the solution (18) may become singular for some combinations of c_1 and the parameters (k, ω) of the equation. This happens when the initial condition fulfills the inequality:

$$|c_1| \leq \frac{(2n+1)kt}{2^{2n}}\left[\sum_{j=0}^{n}\binom{2n+1}{j}\mathrm{sinc}((2(n-j)+1)\omega t)\right] \, . \tag{19}$$

These oscillators remain invariant under the change of sign of ω, which is due to commutativity of the factorization operators and also the family of q Bernoulli equations and their solutions are invariant under change of sign of ω. For the k parameter, we found the symmetry operation:

$$x(\omega, k, t) = x(\pm\omega, -k, -t). \tag{20}$$

It implies that for negative values of k the trajectories in the phase portrait are a counter clockwise inversion with respect to the positive k. This change in the sign of k turns negative the nonlinear friction coefficient $(q + 2)kx^q$. Despite this, the isochronism of these nonlinear oscillators is not affected.

3 The Odd Power MEOLHT Equations

In this section, we consider the modified Emden equations (1) for $q = 2n + 1$, i.e.,

$$\ddot{x} + k(2n+3)x^{2n+1}\dot{x} + k^2x^{4n+3} + \omega^2 x = 0 \, , \tag{21}$$

more exactly we are concerned with the first four odd cases, $n \in \{0, 1, 2, 3\}$. The arbitrary phase δ is set to zero which provides more compact formulas. In addition, equation (19) simplifies to

$$|c_1| \leq \frac{n!2^n(2n+1)}{(2n+1)!!}\frac{k}{\omega} \, . \tag{22}$$

In the next particular cases, all equations are obtained from the general results (15), (17), (18), (21), and (22).

- For $n = 0$, we obtain the cubic MEOLHT oscillator with linear friction coefficient (8) with the solution

$$x(t) = \frac{\cos(\omega t)}{c_1 + \frac{k}{\omega}\sin(\omega t)} = \frac{\cos(\omega t)}{c_1 + kt\,\mathrm{sinc}(\omega t)}\,. \tag{23}$$

This solution is singular if $c_1 \in [-k/\omega, k/\omega]$, otherwise it is periodic of period $T = 2\pi/\omega$. In Fig. 1, one can see plots of this solution for three different initial conditions, one in the singular regime and two in the bound isochronous range.

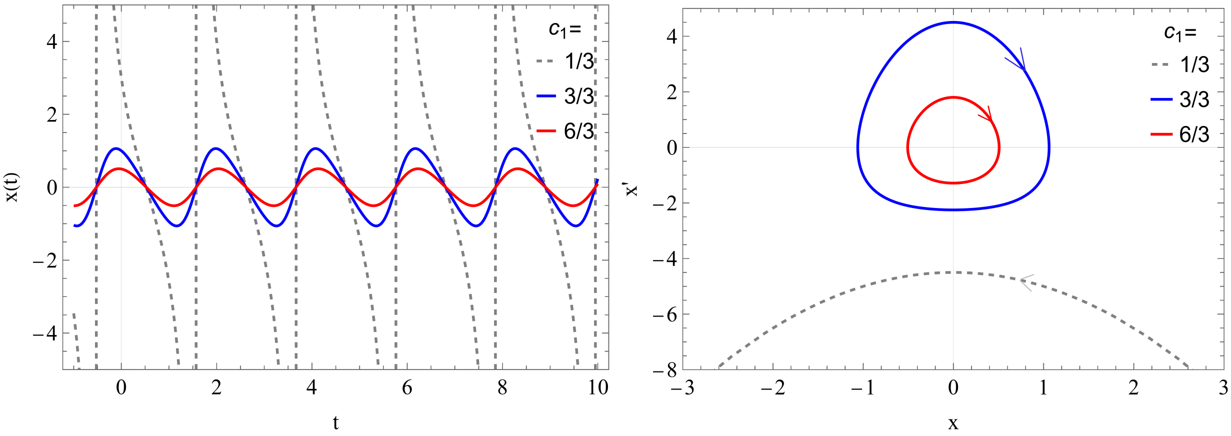

Figure 1: Case $n = 0$ for $k = 1$, $\omega = 3$ and the indicated values of c_1. The figures in column 1 correspond to solutions (23) of the n=0 case, while those in column 2 show their phase portraits. The closed trajectories in the phase portrait are deformed ellipses for the non singular cases and they are clock-wise oriented which is due to the positive sign of k.

- For $n = 1$, we obtain the seventh power MEOLHT oscillator with cubic friction coefficient

$$\ddot{x} + 5kx^3\dot{x} + k^2x^7 + \omega^2 x = 0 \tag{24}$$

related to the Bernoulli equation

$$\dot{x} + \omega\tan(\omega t)x = -kx^4 \tag{25}$$

with solution

$$x(t) = \frac{\cos(\omega t)}{[c_1 + kt\left(\frac{9}{4}\,\mathrm{sinc}(\omega t) + \frac{1}{4}\mathrm{sinc}(3\omega t)\right)]^{1/3}}\,. \tag{26}$$

The denominator of this solution presents only oscillatory terms, and the dynamics is related to periodic motion with closed orbits around a center. The biggest argument in the sinc function corresponds to the $n+2$ value. The solution is not singular if $c_1 \notin [-2k/\omega, 2k/\omega]$. Plots of this case are displayed in Fig. 2.

- For $n = 2$, we obtain the eleventh power MEOLHT oscillator with a fifth power friction coefficient

$$\ddot{x} + 7kx^5\dot{x} + k^2x^{11} + \omega^2 x = 0 \tag{27}$$

related to the Bernoulli equation

$$\dot{x} + \omega\tan(\omega t)x = -kx^6 \tag{28}$$

with solution

$$x(t) = \frac{\cos(\omega t)}{[c_1 + kt\left(\frac{50}{16}\mathrm{sinc}(\omega t) + \frac{25}{16}\mathrm{sinc}(3\omega t) + \frac{5}{16}\,\mathrm{sinc}(5\omega t)\right)]^{1/5}}\,. \tag{29}$$

The sinc functions in the denominator are of consecutive odd harmonics with the fifth the highest one. Plots of this case are displayed in Fig. 3. The singularity range of this solution is $c_1 \in [-8k/3\omega, 8k/3\omega]$, while for c_1 outside it the solution is periodic of period $T = 2\pi/\omega$.

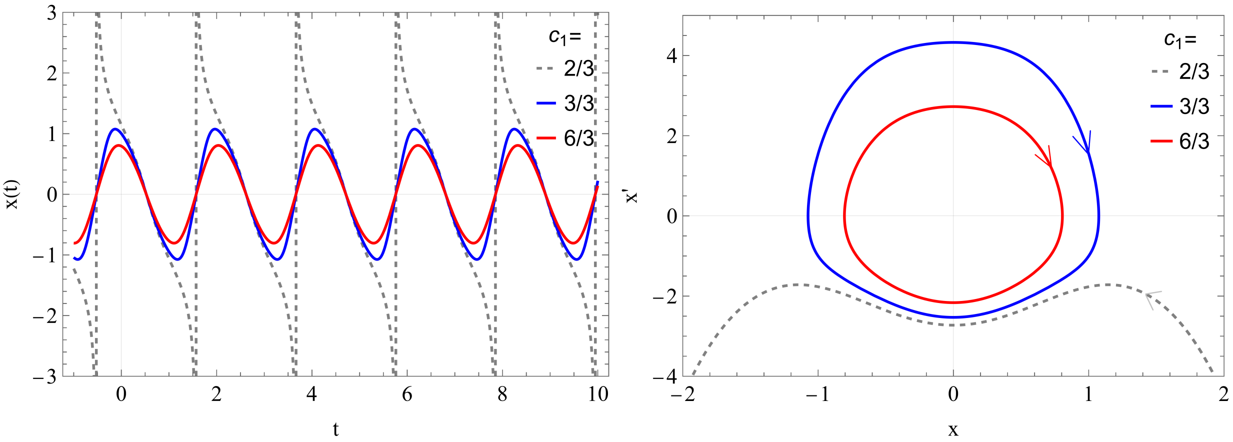

Figure 2: Plots with the same k and ω parameters as in the previous figure, but for $n = 1$. In column 1, Solutions (26) of the $n = 1$ MEOLHT case are shown, while column 2 presents the phase portraits of the $n = 1$ solutions.

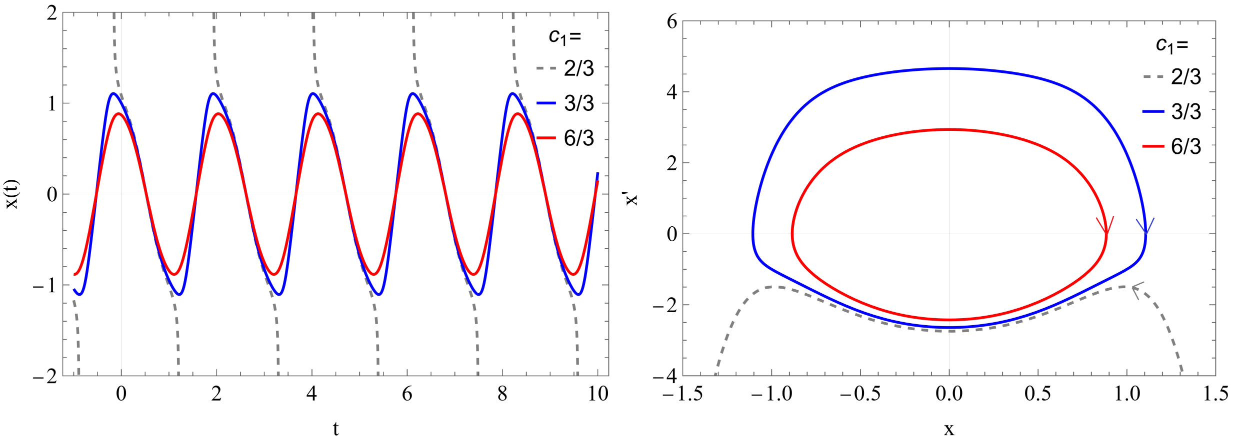

Figure 3: Plots with the same k and ω parameters as in the previous figures, but for $n = 2$. In column 1, solutions (29) of the $n = 2$ MEOLHT case are shown, while column 2 presents the phase portraits of the $n = 2$ solutions.

- For $n = 3$, we obtain the fifteenth power MEOLHT oscillator with a seventh power friction coefficient

$$\ddot{x} + 9kx^7\dot{x} + k^2x^{15} + \omega^2 x = 0 \tag{30}$$

related to the Bernoulli equation

$$\dot{x} + \omega\tan(\omega t)x = -kx^8 \tag{31}$$

with solution

$$x(t) = \frac{\cos(\omega t)}{\left[c_1 + kt\left(\frac{245}{64}\mathrm{sinc}(\omega t) + \frac{147}{64}\mathrm{sinc}(3\omega t) + \frac{49}{64}\mathrm{sinc}(5\omega t) + \frac{7}{64}\,\mathrm{sinc}(7\omega t)\right)\right]^{1/7}}\,. \tag{32}$$

For this case, which is plotted in Fig. 4, the sequence of sinc functions in the denominator of (32) contains the odd harmonics up to the seventh one. The singularity interval of the solution increases to $c_1 \in [-16k/5\omega, 16k/5\omega]$, whereas the bounded solutions are periodic of period $T = 2\pi/\omega$.

An important feature of these oscillators is that the forbidden range in k/ω to have bounded isochronous solutions becomes bigger and bigger at higher orders of nonlinearity tending asymptotically to cover the whole k/ω real line.

4 Conclusion

The modified Emden nonlinear oscillators with additional linear harmonic term of arbitrary positive odd order $q = 2n + 1$ can be solved in closed form by the equivalent Bernoulli equations which are well-known linearizable equations. The reduction is achieved via mathematical manipulations within the generalized commutative factorization method. The cases $n = 0, 1, 2, 3$ ($q = 1, 3, 5, 7$) are presented as direct applications of the general results obtained

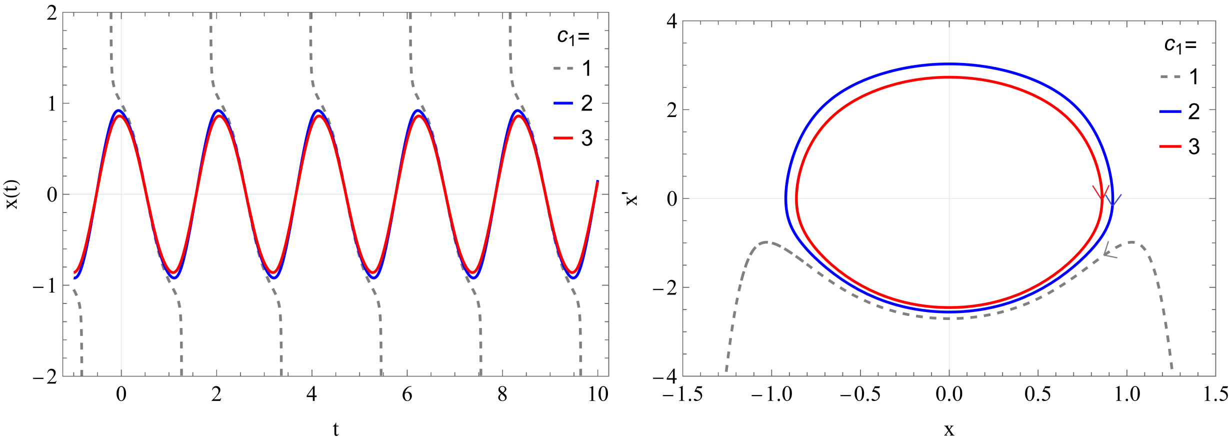

Figure 4: Case $n = 3$ for $k = 1$, $\omega = 3$ and the indicated values of c_1. In column 1, solutions (32) of the $n = 3$ MEOLHT case are shown, while column 2 presents the phase portraits of the $n = 3$ solutions.

for arbitrary odd order q. For the regular solutions, we show that this system has a center at the origin which does not depend on the values of k, q, ω, or the initial conditions. In the case of even order $q = 2n$, these oscillators are not isochronous, they are instead non exponentialy underdamped. Their study is work in progress that we hope to accomplish in the near future.

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