

Convex Hull as a Method for Characterizing Chaotic Systems

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Abstract

This work introduces a geometric-quantitative characterization of chaotic attractors based on the convex hull of their phase-space representation. In contrast to the classical dynamical characterization — based on time-series descriptors such as Lyapunov exponents, Kaplan-Yorke and fractal dimensions, which quantify the temporal signature of chaos — the proposed characterization treats the attractor as a spatial object and quantifies its global extent through the area of its 2D projections and its total volume in phase space. The methodology is applied to two families of chaotic systems: three variants of the Chua oscillator sharing the double-scroll topology, and a piecewise-linear jerk system generating a nine-scroll attractor under integer- and fractional-order dynamics. In every case, the classical descriptors confirm that the systems remain qualitatively equivalent in their chaotic behavior, while the convex-hull descriptors reveal deformations of one to two orders of magnitude that the classical metrics overlook. We formalize this observation as a proposition establishing the independence of the geometric descriptors from the classical dynamical ones. The proposed characterization is complementary rather than alternative to the classical one, and provides a criterion to discriminate between chaotic systems precisely in the regime where topology, Lyapunov spectrum, and fractal dimensions are inconclusive, a regime particularly relevant for engineering applications where amplitude constraints are decisive.

Keywords— Chaotic attractor, chaos, dynamical systems, fractal dimension, convex hull.

1 Introduction

The history of chaos is not a history of mere randomness, but of hidden patterns in complexity, of deterministic systems whose trajectories are so sensitive to initial conditions that their long-term behavior seems profoundly unpredictable. For centuries, this phenomenon has been observed only anecdotally (in unpredictable swirls of smoke, unpredictable drips of a faucet, or grand, unruly movements of celestial bodies) and was dismissed as either too complicated to quantify or too insignificant to study [1, 2, 3].

The search for an understanding of chaos often shifted from the abstract world of mathematics to the tangible world of physics. For decades, strange attractors like those of Lorenz and Rössler were mathematical phantoms, observed in computer simulations of differential equations, but never actually put into practice. They described the weather and chemical reactions— complex, diffuse systems that are difficult to control or isolate. The Chua attractor story is about how chaos was finally tangible, engineered, and clearly manifested in a simple, predictable way via a simple electronic circuit. It is the chronicle of a theoretical prediction that became a physical reality and revolutionized the study of nonlinear dynamics by serving as a perfect laboratory specimen [4].

The story begins not in a laboratory, but at the University of California, Berkeley, in the mind of an electrical engineer named Leon O. Chua. In the early 1980s, Chua was a leading figure in non-linear circuit theory. He had previously formulated a theoretical basis for a new, simpler type of chaotic circuit. Chua's key insight was a conjecture: that a chaotic attractor, the most complex behavior a dynamical system can exhibit, can be generated by an autonomous circuit containing only three simple, standard components: resistors (R), capacitors (C), and inductors (L), combined with a single non-linear element [5, 6, 7, 8]. This nonlinear element was Chua's own invention: the Chua diode. It was not a typical diode but an active circuit element with a precisely designed, piecewise-linear

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current-voltage characteristic. Its curve resembled a zigzag line with three segments. This deliberate, simplified non-linearity was the decisive factor. Unlike the complex, smooth nonlinearities found in fluid systems or population models, Chua's diode was designed to be mathematically tractable. In a seminal 1986 paper, Chua and his student T. Matsumoto theorized this circuit and proved that it should exhibit chaos. They provided the mathematical equations and the computer-simulated attractor, a double-scroll structure that appeared to orbit around two distinct wells.

Remark 1.0.1. The anecdote about the first time the Chua circuit was connected says that when they set everything up and turned on the power, the output of the circuit should have shown a periodic oscillation or a static voltage. Instead, the oscilloscope screen showed a persistent, dynamic, and never-repeating pattern—a chaotic signal. They adjusted the knobs on the variable resistor and the pattern changed, going through a series of dynamical behaviors: from stable points to periodic cycles to the characteristic chaotic double-scroll attractor. At that moment, the abstract mathematical phantom of chaos became a physical and electronic reality. The Chua circuit was born and the Chua attractor was revealed to the world.

The decisive moment of confirmation came not from Chua's own laboratory but from a collaboration on the other side of the globe. In 1984, a young Japanese researcher, Toshimitsu Matsumoto, spent a year in Berkeley working with Chua. On his return to Waseda University in Tokyo, he commissioned his doctoral student, Makoto Komuro, to build the Chua circuit, which he realized and thus the Chua circuit was born [9]. The significance of this achievement cannot be overestimated. The researchers had a robust, affordable, and completely predictable chaotic system on their workbench for the first time. It was autonomous (it required no external signal input) and low-dimensional (only three equations of state governed it), making it mathematically simple to analyze compared to the infinite-dimensional chaos of turbulence. The circuit was so simple that undergraduate students could build it, which is still being built today. It was a milestone of chaos theory; a simple model organism that made endless experiments possible.

This physical prototype provided an unprecedented testing ground for the tools used to characterize chaos. The Chua attractor became the standard object for testing computational methods.

- **Lyapunov exponents** [10, 11]: The circuit equations enabled for the precise calculation of its Lyapunov spectrum. The researchers confirmed that it has the canonical signature of chaos: $(+, 0, -)$. A positive exponent confirms the exponential divergence of the trajectories, a zero exponent corresponds to the direction along the flow, and a negative exponent ensures that the trajectories remain confined on the attractor. This was no longer just an output of a weather model, but a measurable electronic fact.
- **Fractal dimension** [12, 13]: The intricate scroll-like structure of the attractor seen on the oscilloscope was a clear candidate for fractal analysis. Using algorithms such as the box-counting dimension, the researchers calculated the fractal dimension of the data captured by the circuit. It was a fractional value between 2.0 and 2.2 for a standard setup. This confirmed that this physical object, generated by a real electronic circuit, had the complex, self-similar structure of a mathematical strange attractor.
- **Kaplan-Yorke Dimension** [12, 14, 15]: The Chua circuit became a prime example of the deep connection between dynamics and geometry. Using the calculated Lyapunov exponents ($\lambda_1 > 0$, $\lambda_2 = 0$, $\lambda_3 < 0$), it was possible to apply the Kaplan-Yorke formula: $D_{KY} = 2 + (\lambda_1 / |\lambda_3|)$. The result agreed consistently with the directly measured fractal dimension. It provided an experimental confirmation of the assumption that a system's dynamic properties (its stretching and folding rates) directly dictate its attractor's geometric structure.

The historical development of chaos theory, from Poincaré's intuitions to the Lorenz computer to Mandelbrot's fractals and the experimental verification of the Chua circuit [11, 16, 17, 18, 19, 20], shows a fundamental path: The field matured through the development of tools to quantify the unquantifiable. The Lyapunov exponent provided the final temporal signature and defined chaos by its evolutionary behavior. Fractal geometry provided the language to describe the complex spatial structure. The Kaplan-Yorke dimension then provided a bridge between these two worlds, showing that dynamics and geometry are two sides of the same chaotic coin.

Traditional metrics for analyzing chaotic dynamical systems focus on time series and are useful to highlight characteristic aspects of chaotic behavior. However, these tools often fail to reveal subtle changes in the overall geometry of the attractor. In many cases, they are very computationally intensive, depend strongly on the quality and length of the available data, and are not always comparable to systems whose attractors have similar topological features, since the extrapolation of information limits itself to confirming the chaotic nature. Convex hull techniques offer an alternative and promising perspective. By constructing the convex hull of the projection of an attractor in phase space, one can capture geometrical changes that reflect differences in the dynamical responses of the system. The area or volume of the convex hull becomes a global quantitative descriptor of the spatial extent of the attractor and provides a complementary view to classical measurements.

This approach based on the convex hull has been shown that fractional-order chaotic systems can undergo remarkable geometric deformations. For example, the convex hull of the Rössler and Álvarez-Curiel attractors shrinks by up to 90%. Such drastic changes go unnoticed if one relies only on indicators such as the Lyapunov exponent or the fractal dimension during characterization [21].

Building on this perspective, we introduce in this work what we call a *geometric-quantitative characterization* of chaotic attractors. The classical characterization of chaos is essentially *dynamical*: it describes the temporal signature of the attractor through descriptors computed from the flow, such as Lyapunov exponents, Kaplan-Yorke and fractal dimensions, and it answers the question of *how* trajectories evolve in time. The characterization proposed here is essentially *spatial*: it treats the attractor as a bounded region of phase space and quantifies *how much* of that space the attractor actually occupies, through the area of its 2D projections and its total 3D volume, computed via the convex hull. The two characterizations are not competing, they answer different questions. The dynamical one certifies that a system is chaotic and identifies its topological class; the geometric one distinguishes between chaotic systems that belong to the same class but implement it with different amplitudes, different nonlinearities, or different derivative orders. The central hypothesis of this work is that the geometric-quantitative characterization is non-redundant with respect to the classical one: two attractors can match arbitrarily closely in Lyapunov exponents, Kaplan-Yorke and fractal dimensions while differing by orders of magnitude in their convex-hull descriptors. This hypothesis is tested empirically on the Chua family in Section 4.1 and on a piecewise-linear jerk system in Section 4.2, and formalized as Proposition 5.1 in Section 5.

We address the following specific questions: are all the Chua systems, or their variants, geometrically equivalent when they are dynamically indistinguishable? Does the same phenomenon persist under fractional-order dynamics and piecewise-linear switching laws? Can the geometric discrepancies be turned into a quantitative selection criterion for engineering applications where the shape of the attractor, and not only its temporal unpredictability, is a design constraint?

The rest of the article is organized as follows. Section 2 introduces the family of Chua systems that motivate the analysis, and Section 3 formulates the problem. Section 4 develops the convex-hull methodology and applies it to the Chua family (Section 4.1) and to a piecewise-linear fractional-order jerk system (Section 4.2). Section 5 formalizes the findings as Proposition 5.1, discusses their implications, and concludes the article.

2 Preliminaries

The Chua attractor has gained great popularity for many reasons, as it was one of the first physical systems in which chaos was mathematically demonstrated [7, 8]. It is also an example of Shilnikov chaos (homoclinic orbits) [22, 23], and unlike Lorenz or Rössler [24, 25], Chua's attractor uses non-smooth nonlinearity, making it analytically tractable, yet chaotic. Chua's attractor remains a cornerstone of chaos theory due to its simplicity, robustness, and rich dynamics. For this reason, different versions and approximations of the same attractor have been described in the literature. The researchers asked *what if we change the non-linearity*? The original piecewise-linear function $f(x)$ was replaced by smoother alternatives, such as a cubic polynomial term [26, 27], or by more elaborate constructions, including higher-order polynomial expansions, sinusoidal switching laws, and hysteretic operators [28, 29, 30]. This led to the discovery of a whole family of strange attractors within the same circuit frame.

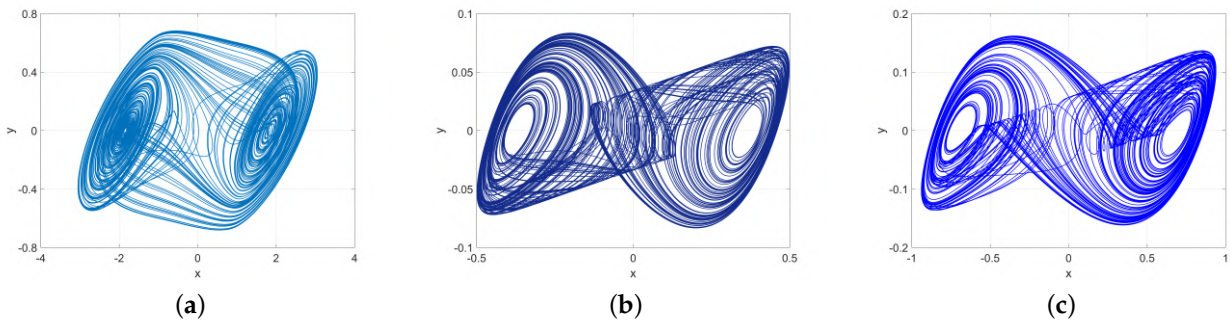


Figure 1: Projection onto the xy - plane of the characteristic attractor of the Chua's systems described in Table 1; (a) Original Chua system, Eq. (1); (b) Cubic Chua system, Eq. (2); (c) Cubic-7 Chua system, Eq. (3).

Table 1: Different representations of the Chua system.

Chua System	Equations
Original [6]	$\begin{aligned}\dot{x} &= \sigma(y - x - f(x)), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y, \\ f(x) &= bx + \frac{1}{2}(a - b)(x + 1 - x - 1).\end{aligned}\tag{1}$
Cubic [26]	$\begin{aligned}\dot{x} &= \sigma y - \sigma x^3 - \sigma c x, \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y.\end{aligned}\tag{2}$
Cubic-7 [27]	$\begin{aligned}\dot{x} &= \sigma \left[y + \frac{x - 2x^3}{7} \right], \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y.\end{aligned}\tag{3}$

Table 2: Parameter values for the Chua system described in Table 1, and attractors displayed in Figure 1.

	Original	Cubic	Cubic-7
σ	10	10	9.5
β	14.87	16	100/7
a	-1.27	—	—
b	-0.68	—	—
c	—	-0.143	—

3 Problem Statement

The different systems and attractors described in Table 1 and Figure 1 have a similar shape, while the characteristic double-scroll attractor of the Chua circuit is preserved. In turn, the systems exhibit similar values in the classical metrics for chaotic systems based on the study of the time series: a positive Lyapunov exponent (close to 0.11), a Kaplan-Yorke dimension around 2.2, and a fractal dimension close to 2.1 [7, 8, 26, 27]. The values may vary between the systems if the parameters or resolution methods are changed but are similar, confirming that the different systems exhibit qualitatively similar chaotic behaviors.

However, a look at the axes in Figure 1 reveals considerable variations in the amplitudes of the different attractors. These changes are overlooked by the classical metrics used to study chaotic systems, but are clearly visible when the three attractors are plotted simultaneously. The questions arise: *Are the geometric changes exhibited by chaotic systems negligible in the sense that they can be safely disregarded once the chaotic nature of the attractor has been confirmed by classical metrics? Do classical metrics for studying chaotic dynamics—namely, Lyapunov exponents, Kaplan-Yorke and fractal dimensions—allow meaningful comparisons between systems that share a common topological structure but differ in their nonlinearities or parameters?* Inspired by previous results where the dynamical variations of the attractors are studied [21, 31], this work addresses these questions by studying the geometric variations in the convex hulls of two families of chaotic systems: three representations of the Chua attractor and a piecewise-linear jerk system under integer- and fractional-order dynamics. Under the premise of considering the geometry of the chaotic attractor instead of the temporal behavior, this work aims to visualize the dynamical changes between systems that share a common topological structure but are qualitatively distinguishable only through their geometry — changes that are generally omitted or invisible using classical metrics of chaotic systems, such as the Lyapunov exponent or the fractal dimension.

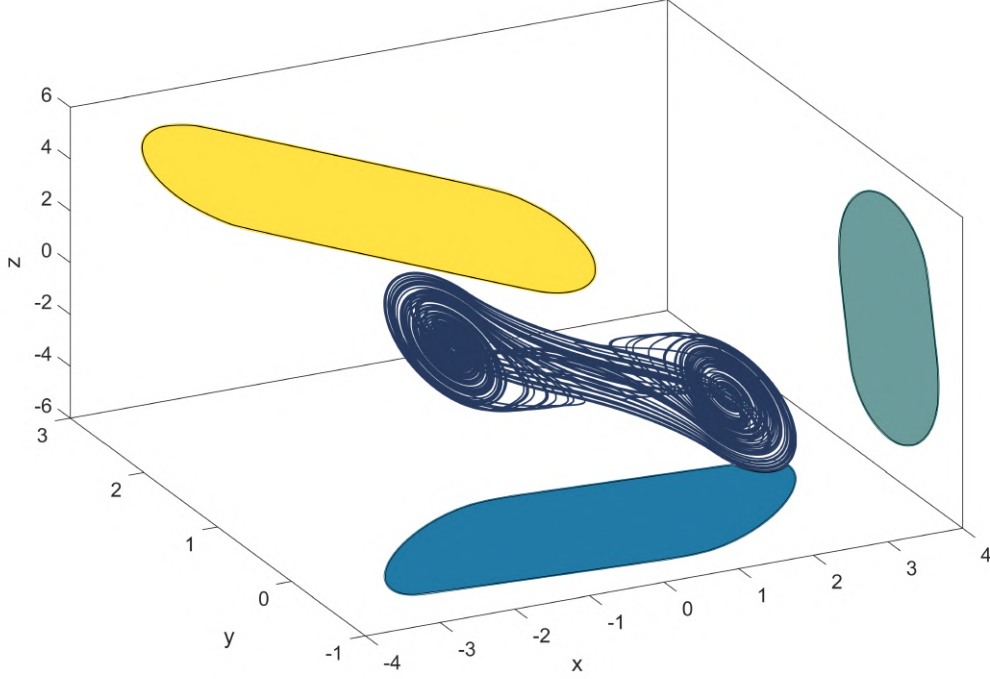


Figure 2: Visualization of the 2D projection surfaces of the Chua attractor obtained with the convex hull for Eq. (1). The areas calculated correspond to the filled regions.

4 Methodology and Results

This study focuses on analyzing geometric variations in chaotic attractors. Using convex hulls, we can enclose the smallest possible convex shape around a 2D projection of an attractor or approximate its volume in a 3D space. This approach provides a quantitative measure of changes in both the projected area and the total volume of the system.

Definition 4.1. Let $\mathcal{S} = \{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_k\}$ be a finite set of k points in the Euclidean space $\mathbb{R}^n$. The convex hull of $\mathcal{S}$, denoted $\text{conv}(\mathcal{S})$, is the smallest convex region that encloses all points of $\mathcal{S}$. For a set $\mathcal{S}$ in $\mathbb{R}^n$, its convex hull is given by

$$\text{conv}(\mathcal{S}) = \left\{ \sum_{i=1}^k \phi_i \mathbf{p}_i \mid \mathbf{p}_i \in \mathcal{S}, \phi_i \geq 0, \sum_{i=1}^k \phi_i = 1 \right\}, \quad (4)$$

where the coefficients ϕ_i , for $i = 1, 2, \dots, k$, satisfy: (i) non-negativity, $\phi_i \geq 0$; and (ii) normalization, $\sum_{i=1}^k \phi_i = 1$. Together, these two conditions ensure that the linear convex combination in Eq. (4) generates the smallest convex set that contains $\mathcal{S}$.

The convex hull is calculated using integrated software functions that evaluate both the area (for 2D projections) and the volume (for 3D spaces) [32, 33]. The procedure for determining a convex hull is as follows and the results obtained are shown in (see Figure 2):

- **Data structuring:** The phase space points of the attractor are organized in a coordinate matrix.
- **Hull construction:** The convex hull algorithm is applied to extract the geometric properties of the enclosing shape.
- **Area calculation:** The projected area of the attractor is calculated for each 2D plane. The result is given in arbitrary square units.
- **Volume estimation:** The total volume occupied by the attractor in the state space is determined. The results are given in arbitrary cubic units.

Remark 4.1.1. Accurate calculations of the convex hull require careful exclusion of transition states from oscillator dynamics. Including transient states can inflate the area and volume estimates and produce artificially larger values. This work performed all analyses after discarding transient periods to ensure reliable results.

4.1 The Chua family

Using the attractors shown in Figure 1, the following values are calculated: (a) the Kaplan-Yorke dimension (D_{KY}), (b) the fractal dimension (D_f) by dimensional reduction using the Poincaré map, and (c) both the volume and the areas of the three projections of the attractor using the convex hull method. All results are shown in Table 3, where A_{jk} denotes the areas in the projection jk of the attractor, and V is for the total volume.

Table 3: Characterization of the Chua attractors described in Table 1 and shown in Figure 1.

	D_f	D_{KY}	$A_{xy}u^2$	$A_{xz}u^2$	$A_{yz}u^2$	Vu^3
Original	2.05	2.0603	6.2606	20.2192	10.0966	11.5953
Cubic-7	2.001	2.0345	0.4920	1.4207	0.7105	0.2354
Cubic	2.001	2.0350	0.1366	0.4166	0.1999	0.0515

Remark 4.1.2. Two clarifications are in order regarding the classical descriptors reported in Table 3. First, the fractal dimension D_f was estimated after reducing the system dimension via the Poincaré section, which yields $D_f^* = D_f - 1$ as described in [34]. The values reported in Table 3 already incorporate this summation factor, and are consistent with the Kaplan-Yorke dimension D_{KY} obtained directly from the Lyapunov spectrum. Second, and more importantly, for the aims of this work, it should be emphasized that the Poincaré section and the convex hull address complementary but distinct questions about the attractor. The Poincaré section is a *dimensional-reduction* tool: by intersecting the flow with a transversal hyperplane, it converts the continuous-time dynamics into a discrete map whose invariant set carries the fractal structure of the attractor. It therefore reveals *local, recurrence-based* information about how trajectories fold onto themselves. The convex hull, by contrast, is a *global, spatial-extent* descriptor: it encloses the attractor in the smallest convex region of the phase space, and quantifies how much of that space the attractor actually occupies. The two descriptors are neither redundant nor interchangeable. Two attractors can share essentially identical Poincaré-derived fractal dimensions, as is the case for the three Chua systems in Table 3, while differing by more than two orders of magnitude in convex-hull volume. In this sense, the convex hull provides information that is genuinely independent of, and complementary to, the Poincaré-based characterization.

4.2 Extension to a piecewise-linear fractional-order multi-scroll system

To assess the generality of the proposed methodology beyond the Chua family, we now apply it to a multi-scroll generator based on the jerk equation with a Saturated Non-Linear Function (SNLF), previously reported in [31]. This system was selected for four reasons: (i) it belongs to the jerk class, which is structurally different from the Chua family; (ii) its nonlinearity is a piecewise-linear (PWL); (iii) it admits both integer- and fractional-order integration, allowing us to probe whether the geometric metrics detect the effect of the fractional operator; and (iv) it generates a 9-scroll attractor whose global shape is known to depend on which state variable carries the fractional-order operator. The system is described by:

$$\begin{aligned} D^{q_x}x &= y, \\ D^{q_y}y &= z, \\ D^{q_z}z &= -\alpha[x + y + z - f(x)], \end{aligned} \tag{5}$$

where D^{q_i} denotes the Caputo fractional-order operator applied to state $i \in \{x, y, z\}$, $\alpha \in (0, 1)$ is the control parameter that places the system in the Unstable Dissipative System type I (UDS-I) region, and $f(x)$ is the SNLF that determines the equilibrium points around which the scrolls are organized. The explicit construction of $f(x)$, together with the electronic implementation of the fractance of order $q = 0.96$, is reported in [31] and is not reproduced here. The time series analyzed in this section were obtained from the electronic-circuit simulation of Eq. (5) in the Multisim development platform, following the implementation reported in [31]. Circuit-level simulation captures effects — such as operational amplifier saturation, component tolerances and the frequency response of the physical fractance — that are not present in a direct numerical integration of the fractional-order equations, and therefore provides an intermediate validation between pure numerical solutions and hardware experiments. We fix $\alpha = 0.45$ and analyze the five integration-order scenarios reported in [31]:

- $S_1 = [1, 1, 1]$ — integer-order (reference).
- $S_2 = [0.96, 0.96, 0.96]$ — commensurate fractional order.

- $S_3 = [1, 1, 0.96]$ — fractional order applied only to z .
- $S_4 = [1, 0.96, 1]$ — fractional order applied only to y .
- $S_5 = [0.96, 1, 1]$ — fractional order applied only to x .

Across all five scenarios, the system remains chaotic: the largest Lyapunov exponent retains the canonical signature of chaos, the fractal dimensions vary only within a narrow range. However, as summarized in Table 4, the convex-hull-based metrics reveal deformations of the attractor that the classical descriptors overlook.

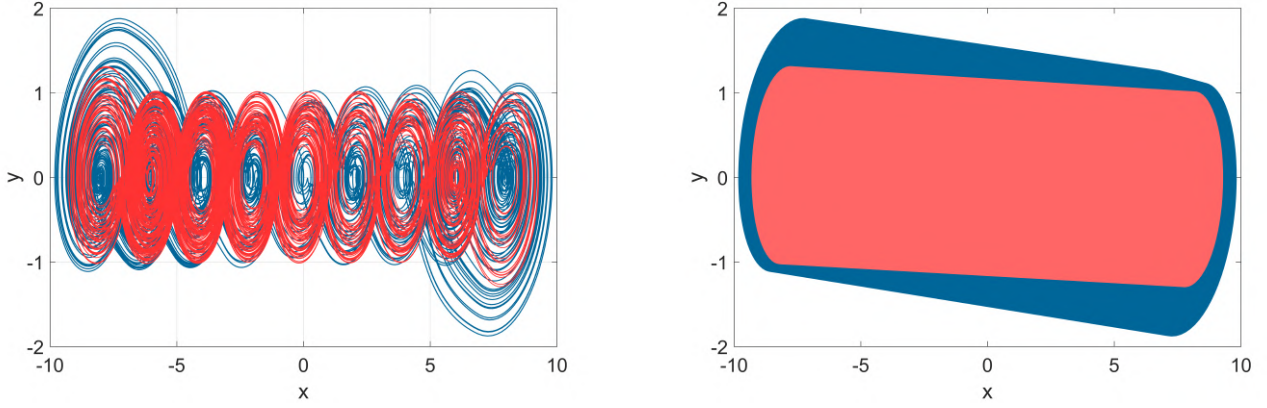


Figure 3: Convex-hull characterization of the SNLF-jerk attractor for scenarios S_1 (integer-order reference) and S_3 (fractional order applied to the z state variable). The two chaotic trajectories are shown in the left panel; the filled regions on the right panel correspond to the convex-hull projections onto the xy plane. Blue: S_1 , with $A_{xy} = 57.39$, $A_{xz} = 34.79$, $A_{yz} = 7.01$, and total volume $V = 81.25$. Red: S_3 , with $A_{xy} = 41.78$, $A_{xz} = 29.87$, $A_{yz} = 4.27$, and total volume $V = 55.91$. The comparison shows that applying the fractional-order operator to the z state variable alone induces a contraction of approximately 31% in the phase-space volume, while the classical dynamical descriptors remain essentially unchanged.

Table 4: Geometric characterization of the 9-scroll SNLF-jerk attractor described by Eq. (5) for scenarios S_1 – S_5 at $\alpha = 0.45$. A_{jk} denotes the convex-hull area in the jk -projection of the attractor, and V is the total convex-hull volume in phase space. All values are given in arbitrary units of A^2 (areas) and V^3 (volume), following the voltage scale of the electronic implementation.

Scenario	q^T	A_{xy}	A_{xz}	A_{yz}	V
S_1	$[1, 1, 1]$	57.39	34.79	7.01	81.25
S_2	$[0.96, 0.96, 0.96]$	51.24	32.71	5.21	67.53
S_3	$[1, 1, 0.96]$	41.78	29.87	4.27	55.91
S_4	$[1, 0.96, 1]$	59.49	30.79	5.80	73.81
S_5	$[0.96, 1, 1]$	51.75	32.85	6.36	66.17

Two observations follow. First, the choice of which state variable carries the fractional-order operator has a directional, non-monotonic effect on the geometry of the attractor: applying $q < 1$ to z (scenario S_3) contracts the attractor volume by 31% relative to the integer-order reference (S_1), whereas applying $q < 1$ to y (scenario S_4) contracts it by only 9%, and applying $q < 1$ to x (scenario S_5) contracts it by 19%. The commensurate case (S_2) yields an intermediate contraction of 17%. This directional effect is not resolved by the classical descriptors, the fractal dimensions across the five scenarios remain within a narrow range, and the Lyapunov exponent analysis retains the canonical chaotic signature, but is clearly captured by the convex-hull descriptors.

Second, the geometric changes induced by the fractional operator are illustrated in Figure 3, which compares S_1 and S_3 : the projection on the xy plane shrinks visibly, and Table 4 confirms that the same effect is present in the other two coordinate planes.

The result is significant in two additional respects. First, because $f(x)$ is a PWL switching function, the analysis confirms that the convex-hull methodology remains informative for attractors generated by non-smooth vector fields, and not only for the smooth (or partially smooth) variants of the Chua system considered in Section 4.1. Second, because scenarios S_2 – S_5 involve fractional-order dynamics, the analysis shows that the methodology captures geometric deformations induced by the memory of the fractional operator, a class of variations that is otherwise difficult to quantify using time-series-based descriptors alone. A broader validation of the geometric perspective across additional attractor families, including the Rössler oscillator, its PWL variant, and additional multi-scroll generators, analyzed under a topological refinement of the convex hull based on the α -shape, is reported in [21, 35].

5 Discussion and Conclusions

Two families of chaotic systems have been examined through the convex-hull methodology in Sections 4.1 and 4.2: three variants of the Chua oscillator sharing the double-scroll topology, and five integration-order scenarios of a piecewise-linear jerk system generating a nine-scroll attractor. In every case, the classical descriptors — Lyapunov spectrum, Kaplan-Yorke and fractal dimensions — confirmed that the systems remain qualitatively equivalent in their chaotic behavior, while the convex-hull descriptors revealed differences of one to two orders of magnitude in projected areas and total volume. This consistent gap between the two families of descriptors motivates the following formal statement.

Proposition 5.1 (Geometric independence from classical chaotic characterization). *Let S_1 and S_2 be two dissipative continuous-time dynamical systems in $\mathbb{R}^n$, each admitting a bounded strange attractor $\mathcal{A}_i$, and assume that both systems satisfy:*

- (a) Topological equivalence: $\mathcal{A}_1$ and $\mathcal{A}_2$ share the same qualitative structure (e.g., double-scroll, spiral, or multi-scroll of the same order).
- (b) ε -similarity in classical chaos metrics: *there exists $\varepsilon > 0$ such that*

$$|\lambda_1^{(1)} - \lambda_1^{(2)}| < \varepsilon, \quad |D_{KY}^{(1)} - D_{KY}^{(2)}| < \varepsilon, \quad |D_f^{(1)} - D_f^{(2)}| < \varepsilon, \quad (6)$$

where $\lambda_1^{(i)}$, $D_{KY}^{(i)}$, and $D_f^{(i)}$ denote the largest Lyapunov exponent, the Kaplan-Yorke dimension, and the fractal dimension of $\mathcal{A}_i$.

Then the geometric descriptors $V(\text{conv}(\mathcal{A}_i))$ and $A_{jk}(\text{conv}(\pi_{jk}\mathcal{A}_i))$, where π_{jk} denotes the projection onto the x_j - x_k plane, need not satisfy any analogous ε -bound. In particular, the ratio

$$\rho := \frac{V(\text{conv}(\mathcal{A}_1))}{V(\text{conv}(\mathcal{A}_2))} \quad (7)$$

can differ from unity by arbitrary orders of magnitude.

Justification. The claim admits a direct argument based on the different invariance properties of the two families of descriptors. The classical metrics — Lyapunov exponents, Kaplan-Yorke and fractal dimensions — are invariant under smooth invertible changes of coordinates, and depend on the local expansion and contraction rates of the flow rather than on the absolute amplitudes of the state variables. The convex-hull descriptors, in contrast, are amplitude-dependent by construction: the linear rescaling $x \mapsto \sigma x$, with $\sigma > 0$, leaves λ_i , D_{KY} and D_f unchanged while scaling every projected area by σ^2 and the total volume by σ^3 . Hence, two systems whose vector fields differ only in an effective amplitude scale exhibit the property claimed above, and the ratio ρ can be made arbitrarily large or small.

Explicit verifications are provided by Table 3 for the Chua family, where the original and cubic versions share D_{KY} and D_f to within $\varepsilon \approx 0.03$ while exhibiting a volume ratio $\rho \approx 225$, and by Table 4 for the SNLF-jerk family, where scenarios S_1 – S_5 share the same qualitative dynamics but their convex-hull volumes span a range of $\rho \approx 1.45$, with the smallest attractor (S_3 , fractional order applied to z) contracting by roughly 31% in volume relative to the integer-order reference (S_1). The comparatively modest value of ρ in this family reflects a structural difference with respect to the Chua case: the SNLF-jerk scenarios share a common electronic implementation whose amplitude scale is fixed by the plateaus of the saturated non-linear function, so the geometric variability is intrinsically bounded. Even under this constraint, the fractional-order operator induces a volumetric change that is not detected by the classical descriptors. Further evidence covering the Rössler system, its PWL variant, and additional multi-scroll systems is reported in [21, 35]. $\square$

Proposition 5.1 makes explicit that the geometric characterization introduced in this work is not a proxy for the dynamical characterization: it captures a distinct and independent property of the attractor. The convex-hull descriptors are therefore best interpreted as *complementary* rather than *alternative* metrics — they discriminate between chaotic systems precisely in the regime where classical descriptors are inconclusive. This directly answers the two questions posed in the Problem Statement of this work. *Are the geometric changes exhibited by chaotic systems negligible once the chaotic nature has been confirmed?* They are not: the Chua family alone exhibits a volume variation of more than two orders of magnitude while its Kaplan-Yorke and fractal dimensions vary by less than 0.03, and the SNLF-jerk family reproduces the same pattern under fractional-order variations. *Do classical metrics allow meaningful comparisons between systems that share a common topological structure but differ in their nonlinearities or parameters?* They allow us to

confirm that the systems are chaotic and topologically equivalent, but they do not resolve the geometric differences that distinguish one implementation from another.

The complementary use of the convex hull as a fundamental metric to study chaotic systems is proposed to allow a quantitative comparison of different systems. The significant differences observed from the geometry of the systems can be decisive in the choice of one model over another for a particular application, e.g., in the implementation of mechanical systems or in the development of integrated circuits, where amplitude limitations are decisive factors [36, 37]. The addition of studies of chaotic systems from the geometric perspective, as we have done, provides an opportunity to establish criteria for choosing one chaotic system over another when the chaotic behavior is a constant.

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Conflicts of interest

The authors declare that they have no conflicts of interest with respect to this article.

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